Chapter 3
Section 3.1

Recap and Then Some

In Section 2.5 we discussed inverses of functions. In particular, we determined that a function is invertible if and only if it is one-to-one.

Strategy: Let f be a one-to-one (invertible) function. Suppose that $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are two ordered pairs in f. If we suppose that $f(x_1) = f(x_2)$ then what must the relationship between x_1 and x_2 be?

If $f(x_1)=f(x_2)$ then $x_1=x_2$.

Exercise: Show that $j(x) = \frac{x+3}{x-5}$ is invertible and find $j^{-1}(x)$.

$j^{-1}(x) = \frac{3+5x}{x-1}$

Example: Consider the quadratic function $g(x) = x^2 - 6x + 13$ and the function $f(x) = \sqrt{x-4} + 3$.

a) Determine the domain of g and f.

dom $g = \mathbb{R}$ dom $f = [4, \infty)$

b) By completing the square, write g as a member of the transformation family of x^2.

$g(x) = (x-3)^2+4$

c) Show that g is not invertible (one-to-one).

$g(4)=5=g(2)$ so g is not one-to-one.

d) Show that $f(x)$ is invertible and find its inverse.

$f^{-1}(x)=(x-3)^2+4$

e) What is the domain of f^{-1}?

dom f^{-1} = ran $f = [3, \infty)$

Question: How is it possible that g is not invertible, but f is?

The domain of g is larger

Further Question: If we restrict the domain of g to a subset of its former self, then we can make g a one-to-one function **on the new domain.** What should this new domain be?

The new domain should be ran f, which is $[3, \infty)$.

Quadratic Functions

A quadratic function is a function defined by a quadratic equation. Another way of classifying quadratic functions is as the set of functions in the transformation family of $y = x^2$ which are not constant functions (i.e. the set of functions of the form $f(x) = a(x-h)^2 + k$ with a, h and k real numbers with $a \neq 0$ or $\{f(x) : f(x) = a(x-h)^2 + k$ and $a, h, k \in \mathbf{R}$ and $a \neq 0\}$).

Exercise: By completing the square, write $f(x) = x^2 + 6$ as a member of the transformation family of $y = x^2$ and graph $f(x)$.

Since there is no linear term, then we do not have to complete the square.
Therefore $f(x)=x^2+6$ is in ~~vector~~ vertex-form with vertex $(0,6)$.

Graphs of Quadratic Functions

If we consider a quadratic function of the form $f(x) = a(x-h)^2 + k$ then we can classify how the graph of $f(x)$ looks just be consider the constants a, h and k. For instance, if $a > 0$ then $f(x)$ **opens upward**; but if $a < 0$ then the graph **opens downward.** Remember also that h determines the amount of horizontal translation and k determines the amount of vertical translation. Because of this, we say that (h, k) is the **vertex** of $f(x)$ and that $f(x) = a(x-h)^2 + k$ is the **vertex form** of f. We can also determine that f has an **axis of symmetry** on the line $x = h$. **Exercise:** Find the vertex of the following quadratic functions.

a) $f(x) = -2x^2 - 4x + 3.$
$f(x) = -2(x+1)^2 + 5 \quad (-1, 5)$

b) $g(x) = 2x^2 - 4x + 9.$
$g(x) = 2(x-1)^2 + 7 \quad (1, 7)$

Definition: The **minimum value** of a function f is the least value $y = f(x)$ for $x \in dom f$. The **maximum value** of the function f is the greatest value $y = f(x)$ for $x \in dom f$.

The domain of every quadratic function is all real numbers, $(-\infty, \infty)$. The range is determined by the second coordinate of the vertex. If f opens upward then its range is $[k, \infty)$ and k is the minimum value of f. If f opens downward then its range is $(-\infty, k]$ and k is the maximum value of f.

Question: What is the domain and range of f and g in the exercise above?

$dom f = \mathbb{R} = dom g \quad ran f = (-\infty, 5] \quad ran g = [7, \infty)$

Quadratic Inequalities

When solving quadratic inequalities there are two different methods you can use. You can either graph it (which we have seen before) or you can use the **test-point method**. **Exercise:** Use the graphical method for solving quadratic inequalities to solve the following inequalities.

a) $x^2 - x > 6.$ $\quad (-2, 3)$

b) $(x+3)^2 + 2 < 6.$ $\quad (-5, -1)$

Exercise: Use the test-point method to solve the following quadratic inequalities.

a) $2x^2 - 4x - 9 < 0.$ $\quad \left(1 - \frac{1}{2}\sqrt{22}, 1 + \frac{1}{2}\sqrt{22}\right)$

b) $w^2 - 4w - 12 \geq 0.$

$[-2, 6]$

Applications of Maximum and Minimum

Exercise: A ball is tossed straight upward with an initial velocity of 80 feet per second from a rooftop that is 12 feet above ground level. The height of the ball in feet at time t in seconds is given by $h(t) = -16t^2 + 80t + 12$. Find the maximum height above ground level for the ball.

112 ft

Exercise: If 100 m of fencing will be used to fence a rectangular region, then what dimensions for the rectangle will maximize the area of the region?

$W = 25m = L$

$A = 625 \ m^2$

Exercise: If 100 m of fencing will be used to fence three sides of a rectangular region (because one side of the region is enclosed by the side of a house) then what dimensions for the rectangle will maximize the area of the region?

$A = 1250 \ m^2$

<u>Exercise:</u> $j(x) = \dfrac{x+3}{x-5} \implies (x-5)j(x) = x+3$

$$j(x)\cdot x - 5\cdot j(x) = x+3$$

$$j(x)\cdot x - x = 3 + 5j(x)$$

$$x(j(x)-1) = 3+5j(x)$$

$$x = \frac{3+5j(x)}{j(x)-1}.$$

Therefore $j^{-1}(x) = \dfrac{3+5x}{x-1}$.

<u>Example!</u> (a) $\operatorname{dom} g = \mathbb{R} = (-\infty, \infty)$ $\quad \operatorname{dom} f = [4, \infty)$

(b) $g(x) = x^2 - 6x + 13$

$g(x) = x^2 - 6x + 9 - 9 + 13$

$g(x) = (x-3)^2 + 4$

(c) $g(4) = (4-3)^2 + 4 = 5 = (2-3)^2 + 4 = g(2)$

So g is not one-to-one.

(d) $f(x) = \sqrt{x-4} + 3$

$f(x) - 3 = \sqrt{x-4}$

$(f(x)-3)^2 = x-4$

$(f(x)-3)^2 + 4 = x$

Therefore $f^{-1}(x) = (x-3)^2 + 4$.

(e) The domain of f^{-1} is the range of f.

Since $\sqrt{x-4} \geq 0$ then

$\operatorname{ran} f = [3, \infty)$.

Exercise: a) $f(x) = -2x^2 - 4x + 3$

$$\frac{f(x)}{-2} = x^2 + 2x - \frac{3}{2}$$

$$\frac{f(x)}{-2} = x^2 + 2x + 1 - 1 - \frac{3}{2}$$

$$\frac{f(x)}{-2} = (x+1)^2 - \frac{5}{2}$$

$$f(x) = -2(x+1)^2 + 5$$

Vertex of f is $(-1, 5)$.

b) $g(x) = 2x^2 - 4x + 9$

$$\frac{g(x)}{2} = x^2 - 2x + \frac{9}{2}$$

$$\frac{g(x)}{2} = x^2 - 2x + 1 - 1 + \frac{9}{2}$$

$$\frac{g(x)}{2} = (x-1)^2 + \frac{7}{2}$$

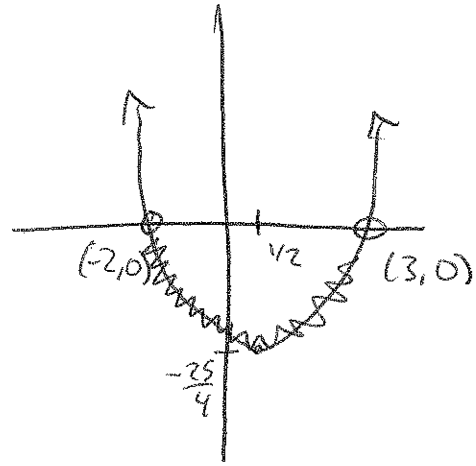
$$g(x) = 2(x-1)^2 + 7$$

Vertex of g is $(1, 7)$.

Exercise: a) $x^2-x-6>0$

$x^2-x+\frac{1}{4}-\frac{1}{4}-6>0$

$(x-\frac{1}{2})^2-\frac{25}{4}>0$



Find zeroes

$x^2-x-6=0$

$(x-3)(x+2)=0$

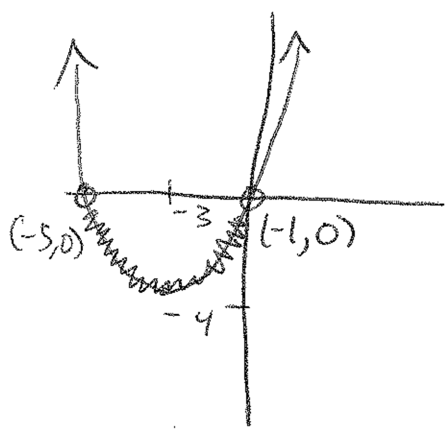
$x=3,-2$. So $x^2-x-6>0$ when

$-2<x<3$

$(-2,3)$.

b) $(x+3)^2+2<6$

$(x+3)^2-4<0$



Find zeroes

$(x+3)^2-4=0$

$(x+3)^2=4$

$(x+3)=\pm 2$

$x=-3\pm 2$

$x=-5,-1$. So $(x+3)^2+2<6$.

when $-5<x<-1$

$(-5,-1)$.

Exercise: a) $2x^2-4x-9<0$

Find zeroes

$2x^2-4x-9=0$

when $x=\frac{4\pm\sqrt{16-4\cdot 2(-9)}}{2\cdot 2}=\frac{4\pm\sqrt{88}}{4}=\frac{4\pm 2\sqrt{22}}{4}=1\pm\frac{1}{2}\sqrt{22}$.

Test a Point in the interval between zeroes.

1 is definitely there.

Is $2(1)^2-4(1)-9<0$?

$-11<0$ Yes.

So $2x^2-4x-9<0$

when $1-\frac{1}{2}\sqrt{22}<x<1+\frac{1}{2}\sqrt{22}$.

$(1-\frac{1}{2}\sqrt{22}, 1+\frac{1}{2}\sqrt{22})$.

b) $w^2-4w-12\geq 0$

Find zeroes

$w^2-4w-12=0$

$(w-6)(w+2)=0$

when $w=6,-2$.

Test a point in between zeroes.

0 is there.

Is $0^2-4(0)-12\geq 0$?

$-12\geq 0$ NO.

So $w^2-4w-12\geq 0$ when

$-2\leq w\leq 6$ $[-2,6]$.

Applications: (1) $h(t) = -16t^2 + 80t + 12$

Write in Vertex Form.

$$\frac{h(t)}{-16} = t^2 - 5t - \frac{3}{4}$$

$$\frac{h(t)}{-16} = t^2 - 5t + \frac{25}{4} - \frac{25}{4} - \frac{3}{4}$$

$$\frac{h(t)}{-16} = \left(t - \frac{5}{2}\right)^2 - \frac{28}{4}$$

$$h(t) = -16\left(t - \frac{5}{2}\right)^2 + 112.$$

Vertex is $\left(\frac{5}{2}, 112\right)$. So maximum of 112 ft.

(2) Perimeter $= 100m = 2W + 2L$

So $100 - 2L = 2W$

$50 - L = W$.

So Area $= L \cdot W = L(50 - L)$

$= -L^2 + 50L$.

Write in Vertex Form.

$$-A = L^2 - 50L$$

$$-A = L^2 - 50L + 625 - 625$$

$$-A = (L - 25)^2 - 625$$

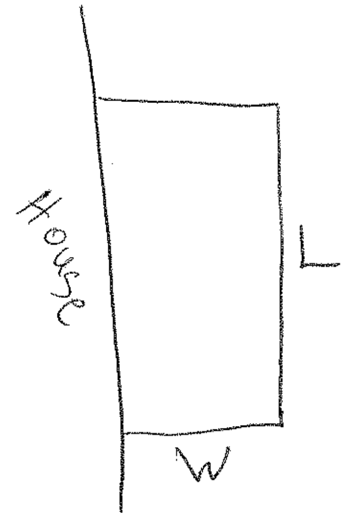
$$A = -(L - 25)^2 + 625$$

Vertex is $(25, 625)$. So maximum of $625\, m^2$.

~~and~~ is obtained when $W = 25$.

Since $50 - L = W$ then $L = 25$.

Applications: (3)



Fence $= 100m = 2W + L$

So $L = 100 - 2W$.

So $\text{Area} = LW = (100-2W)W$

$$A = -2W^2 + 100$$

$$\frac{A}{-2} = W^2 - 50W$$

$$\frac{A}{-2} = W^2 - 50W + 625 - 625$$

$$\frac{A}{-2} = (W-25)^2 - 625$$

$$A = -2(W-25)^2 + 1250.$$

Vertex $= (25, 1250)$. Area is maximized at $1250\,m^2$.

It is obtained when $W = 25$.

Since $L = 100 - 2W$, $L = 50$.